JOURNAL OF FUNCTIONAL ANALYSIS **92**, 474–508 (1990)

Extensions of Positive Definite Integral Kernels on the Heisenberg Group*

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Communicated by the Editors

Received March 18, 1988; revised June 26, 1989

We consider positive definite extensions of functions and distributions which are defined on a cylindrical neighborhood (and positive definite) in the Heisenberg group. When rotational invariance is assumed, we show existence of p.d. extensions. We also give formulas (of the classical Bochner type) for *all* such extensions. They are expanded in terms of an integral transform associated to the sub-Laplacian. We further show that the p.d. extension problem is closely related to the theory of extensions of hermitian infinitesimal representations in the sense considered earlier by Powers and Fuglede. These are extensions to integrable representations in generally larger (so-called *dilation*) Hilbert spaces.

1. INTRODUCTION

It was proved by Krein [26] that a positive definite continuous function on an interval always extends to a positive definite function on the line. But Rudin [38] showed that the corresponding result fails in $\mathbb{R}^2$. It was shown later that extensions exist, in the context of $\mathbb{R}^n$, $n \geqslant 2$, when additional invariance conditions are imposed. We refer to [39, 31, 32]. The extension problem was also considered in [33] for certain semisimple Lie groups. In this paper, we study the extension problem for the Heisenberg group and in a general context as well. Some new results for abelian groups are also included. We restrict attention to the Heisenberg group of three real dimensions, but the proofs carry over easily to the $(2n+1)$-dimensional Heisenberg group, $n = 1, 2, \ldots$.

We shall choose the following convenient coordinates on the Heisenberg group $G := \{(z, x): z \in \mathbb{C},\ x \in \mathbb{R}\}$ with multiplication

$$(z, x) \cdot (z', x') = (z + z', x + x' + 2\,\mathrm{Im}(z\bar{z}')).$$

* Work supported in part by the NSF.

0022-1236/90 $3.00

Let the $\mathbb{T}^1$-action on G be denoted

$$\zeta \cdot (z, x) = (\zeta z, x), \zeta \in \mathbb{T}^1, (z, x) \in G.$$

Let B denote the unit-cylinder in G, $B = \{(z, x) \in G: |z| < 1\}$. We shall also need polar coordinates $g = (r, t, x) \in \mathbb{R}_+ \times (\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}$ corresponding to $z = re^{it}$, $r = |z|$.

A function K on $B \times B$ is said to be *bi-invariant* if there is a function k on the coordinate space $\mathbb{R}_+^2 \times (\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}$ for $G \times G$ such that

$$K(g_1, g_2) = k(r_1, r_2, t_1 - t_2, x_1 - x_2), \tag{1.1}$$

where $g_j = (r_j, t_j, x_j) \in B$, $j = 1, 2$. If K is continuous, we assume that the function k may also be chosen continuous. It is easy to see that K is bi-invariant iff

$$K(\zeta \cdot g_1, \zeta \cdot g_2) = K(g_1, g_2) \tag{1.2}$$

and

$$K(T(s)\, g_1, T(s)\, g_2) = K(g_1, g_2) \tag{1.3}$$

for all $\zeta \in \mathbb{T}^1$, $s \in \mathbb{R}$, and $g_1, g_2 \in B$, where $T(s)$ denotes the translation group in the x-variable. Specifically, $T(s)(z, x) := (z, x + s)$, $s \in \mathbb{R}$, $(z, x) \in G$.

A continuous function K on $B \times B$ is said to be *positive definite* if

$$\int_B \int_B K(g_1, g_2)\, \varphi(g_1)\, \overline{\varphi(g_2)}\, dg_1\, dg_2 \geqslant 0 \qquad \text{for all} \quad \varphi \in C_c^\infty(B), \tag{1.4}$$

where dg denotes the Haar measure on G.

We say that K is *strongly positive definite* if it is further assumed that

$$\int_B \int_B K(g_1, g_2)(L\varphi)(g_1)\, \overline{\varphi(g_2)}\, dg_1\, dg_2 \geqslant 0 \qquad \text{for all} \quad \varphi \in C_c^\infty(B), \tag{1.5}$$

where L denotes the left-invariant sub-Laplacian on G given by (2.5) and (2.8).

It is known that L is a positive self-adjoint operator in $\mathscr{L}^2(G)$ with absolutely continuous (Lebesgue) spectrum, i.e., uniform multiplicity. Moreover, the spectral transform was found in [21, 22], and it is given as follows: For $j, m \in \mathbb{Z}$, $j \geqslant 0$, $|m| \leqslant j$, $\xi \in \mathbb{R}$, define the function

$$h(j, m, \xi, r) = \sqrt{\frac{(j - |m|)!\, 2^{2|m|+1}}{\Gamma(j + |m| + 1)}}\, (|\xi|\, r^2)^{|m|} \times e^{-|\xi| r^2} L_{j-|m|}^{2|m|}(2\, |\xi|\, r^2), \tag{1.6}$$

where L_u^v are the familiar Laguerre polynomials of order v [1].

Let the transform W be defined by

$$W\varphi(j, m, \xi) = (\varphi, h(j, m, \xi, r)e^{i(mt+\xi x)})_{\mathscr{L}^2(G)}$$

for $\varphi \in \mathscr{L}^2(G)$, and where $(\cdot, \cdot)_{\mathscr{L}^2(G)}$ denotes the $\mathscr{L}^2$-inner product on G. This is the spectral transform for the operator L, defined on all of G, in the sense that

$$WL\varphi(j, m, \xi) = \lambda W\varphi(j, m, \xi), \tag{1.7}$$

where $\lambda = (2j+1)\,|\xi| + 2m\xi$. The formula holds in the maximal domain of the operator L consisting of $\varphi \in \mathscr{L}^2(G)$ satisfying $L\varphi \in \mathscr{L}^2(G)$.

It will be convenient to view continuous functions on subsets B of G as distributions on B. A similar note applies to functions on G, and on $G \times G$. The terms defined, so far, for continuous functions extend naturally to distributions. We get the notions of positive definite distributions, positive definite distribution kernels; and the notion of invariance extends readily to distributions. For actions of $\mathbb{T}^1$, and of $\mathbb{R}$, the invariance for distributions is defined relative to the familiar duality between test functions and distributions.

THEOREM 1.1. *Let K be a continuous function on $B \times B$, where B denotes the unit cylinder in the Heisenberg group G. Assume K is bi-invariant and strongly positive definite.*

(a) *Then there is a finite positive Radon measure μ on*

$$\Omega = \{(j, m, \xi) \colon j, m \in \mathbb{Z},\ j \geqslant 0, |m| \leqslant j,\ \xi \in \mathbb{R}\} \tag{1.8}$$

such that

$$k(r_1, r_2, t, x) = \int_\Omega h(\omega, r_1)\, h(\omega, r_2) e^{-i(mt+\xi x)}\, d\mu(\omega), \tag{1.9}$$

where k is the function on $\mathbb{R}^2_+ \times (\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}$ determined by $K(g_1, g_2) = k(r_1, r_2, t_1 - t_2, x_1 - x_2)$ for $g_i = (r_i, t_i, x_i) \in B$, $i = 1, 2$.

(b) *The function on the right-hand side of formula* (1.9) *is positive definite on $G \times G$ and is an extension of the given positive definite function K on $B \times B$.*

(c) *Every function on $B \times B$ which admits an integral representation* (1.9) *for some positive measure μ is positive definite.*

The proof of this result will occupy the next two sections. The first step is a generalized Gelfand–Naimark–Segal (GNS) construction, and we refer to [5, 43, 3, 37] for background material.

We show that a natural Hilbert space $\mathscr{H}_K$ is associated to the given kernel K on $B \times B$. It will be obtained by the familiar quotient construction, and the Hilbert space completion, starting with the space $\mathscr{D} = C_c^\infty(B)$ of compactly supported smooth functions on B. The space $\mathscr{D}$ is a *nuclear space* [45, Chap. 50] when equipped with its familiar *inductive limit topology* of L. Schwartz, cf. [45, Chap. 13]. We then construct a certain abelian von Neumann algebra, and we obtain the integral representation for the given function K by use of direct integral theory relative to the particular abelian von Neumann algebra. We refer to Ref. [30, 3] for direct integral decompositions.

The von Neumann algebra will be generated by a unitary representation of a certain 3-dimensional abelian Lie group, and we apply the Stone–Naimark–Ambrose–Godement (SNAG) theorem in obtaining a generic projection valued measure for the problem. In Section 4 we show that this measure has uniform multiplicity one.

In Section 5, we consider the extension problem on the Heisenberg group for positive definite functions and distributions. This is the special case of the problem for kernels K (i.e., functions or distributions in two variables) such that $K(g_1, g_2) = F(g_1 \cdot g_2^{-1})$, $g_1, g_2 \in G$, where F is a function, or distribution, in a single variable.

We make use of certain Friedrichs extensions in Sections 4 and 4. In Section 6, we give a formula for this extension operator in terms of a heat equation kernel on the Heisenberg group.

In Section 7, we show that the extension problem for functions, and distributions, is closely related to an extension problem for hermitian representations of a $*$-algebra. This $*$-algebra $\mathfrak{A}$ is the universal enveloping algebra [6] of the Lie algebra of the group under study.

If ρ is a representation of $\mathfrak{A}$ in a Hilbert space $\mathscr{H}$, an extension is defined to be a *unitary* representation $(U, \mathscr{K})$, and an isometry S: $\mathscr{H} \to \mathscr{K}$ such that the identity

$$dU(A)S = S\rho(A)$$

holds on the domain of ρ for all $A \in \mathfrak{A}$. Such operator theoretic extensions have been studied earlier by Powers [37], Fuglede [10], Schmüdgen [40, 41], Pedersen [34], and the author [15, 16, 24, 25]. Methods and techniques from this subject apply to the original (classical) extension problem. As a bonus, we also get generalizations of Fuglede's solution [9] to a problem of Segal.

2. An Abelian von Neumann Algebra

In proving Theorem 1.1, we shall use a certain abelian von Neumann algebra which we proceed to describe.

Let $\mathscr{D}$ denote the space of all compactly supported C^∞-functions on the cylinder B equipped with the inductive limit topology, i.e., $\mathscr{D} = C_c^\infty(B)$ with seminorms parametrized by compact subsets of B and by differential operators. We shall define an inner product on $\mathscr{D}$ as follows:

$$(\varphi, \psi) = \int_B \int_B K(g_1, g_2)\, \varphi(g_1)\, \overline{\psi(g_2)}\, dg_1\, dg_2. \tag{2.1}$$

Since K is assumed continuous, and the two functions φ, $\psi \in \mathscr{D}$ are of compact support, the integral is convergent. We have $(\varphi, \varphi) \geqslant 0$ by virtue of the assumption on K.

By the Cauchy–Schwarz inequality we have

$$\mathscr{K} = \{\varphi \in \mathscr{D} : (\varphi, \psi) = 0, \psi \in \mathscr{D}\}. \tag{2.2}$$

We recall the following three operators on $\mathscr{D}$:

$$R(\zeta)\, \varphi(z, x) = \varphi(\zeta z, x), \qquad \zeta \in \mathbb{T}^1, (z, x) \in G, \tag{2.3}$$

$$D\varphi(z, x) = \frac{\partial}{\partial x}\, \varphi(z, x), \tag{2.4}$$

and

$$L\varphi(z, x) = -\frac{\partial^2}{\partial z\, \partial \bar{z}}\, \varphi + iD\left(z \frac{\partial}{\partial z} - \bar{z} \frac{\partial}{\partial \bar{z}}\right) \varphi - |z|^2 D^2 \varphi. \tag{2.5}$$

Points g in the Heisenberg group are parametrized by one complex coordinate z and one real x.

Specifically, we have the left-invariant vector fields

$$Z = \frac{\partial}{\partial z} + i\bar{z}D, \tag{2.6}$$

$$\bar{Z} = \frac{\partial}{\partial \bar{z}} - izD, \tag{2.7}$$

and

$$L = -\tfrac{1}{2}(Z\bar{Z} + \bar{Z}Z). \tag{2.8}$$

We have the following

LEMMA 2.1. *The inner product* $(\cdot,\cdot)$ *defined by the kernel* K *satisfies the invariance conditions,*

$$(R(\zeta)\varphi, R(\zeta)\psi) = (\varphi, \psi), \tag{2.9}$$

$$(D\varphi, \psi) = -(\varphi, D\psi), \tag{2.10}$$

$$(L\varphi, \psi) = (\varphi, L\psi), \tag{2.11}$$

and

$$(L\varphi, \varphi) \geqslant 0 \tag{2.12}$$

for $\varphi, \psi \in \mathscr{D}$, *and* $\zeta \in \mathbb{T}^1$.

Proof. Formula (2.9) is an immediate consequence of (1.2), (2.3), and the formula $dg = r\,dr\,dt\,dx$ for Haar measure Formula (2.10) follows from (1.1).

Formula (2.11) is a consequence of the assumption on K relative to the sub-Laplacian: We may view K as a distribution on $G \times G$. The sub-Laplacian L may be viewed as a differential operator in each one of the two variables g_1, g_2 in the kernel. The distinction between the variables will be indicated with subscripts L_1, L_2, respectively. It follows from (1.5) that $K(g_1, g_2)$ satisfies the differential equation

$$L_1 K = L_2 K \qquad \text{in} \quad B \times B. \tag{2.13}$$

Formula (2.12) now follows from the interpretation of the action of the two operators L_1 and L_2 in the sense of distributions.

Property (2.11) follows from the positivity assumption (1.5) above: Consider the quadratic form Q defined on $\mathscr{D} = C_c^\infty(B)$ by

$$Q(\varphi, \psi) = \int_B \int_B K(g_1, g_2)\, L\varphi(g_1)\, \overline{\psi(g_2)}\, dg_1\, dg_2. \tag{2.14}$$

Then (1.5) and (2.12) are equivalent to the positivity, $Q(\varphi, \varphi) \geqslant 0$, while (2.11) is equivalent to reality of the quadratic form, $Q(\varphi, \varphi) = (L\varphi, \varphi)$. But positivity implies reality, and we conclude that (1.5) implies (2.11).

LEMMA 2.2. (a) *The operators* $\{R(\zeta) \colon \zeta \in \mathbb{T}^1\}$, D, *and* L *from Lemma* 2.1 *pass to the quotient* $\mathscr{D}/\mathscr{N}$, *and define operators on the Hilbert space completion* $\mathscr{H}_K$ *of this quotient.*

(b) *Then*

(i) $\{R(\zeta) \colon \zeta \in \mathbb{T}^1\}$ *is a unitary one-parameter group on* $\mathscr{H}_K$;

(ii) D *is essentially skew-adjoint as an operator in* $\mathscr{H}_K$;

(iii) *If L_Q denotes the Friedrichs extension of the densely defined semidefinite operator L in $\mathscr{H}_K$, then the triple operator family consisting of $\{R(\zeta): \zeta \in \mathbb{T}^1\}$, D, and L_Q is commutative in the strong sense; i.e., the spectral projections of the respective operators generate an abelian von Neumann algebra.*

Proof. From (2.2) it follows that the four operators $T(s)$, $R(\zeta)$, $\zeta \in \mathbb{T}^1$, D, and L pass to the quotient and, moreover, that $R(\zeta)$ passes to a unitary one-parameter group on $\mathscr{H}_K$ relative to the given quadratic form (2.1). The argument for D, and for L, is the same: If $\varphi \in \mathscr{K}$, then $(D\varphi, \psi) = -(\varphi, D\psi) = 0$ for all $\psi \in \mathscr{D}$. It follows that D passes to a linear operator on $\mathscr{D}/\mathscr{K}$. We must check that this operator is essentially skew-adjoint with dense domain $\mathscr{D}/\mathscr{K}$ in the Hilbert space $\mathscr{H}_K$.

When the formula for the inner product (φ, ψ) is written out, we get

$$(\varphi, \psi) = \iint K(r_1, r_2, t_1 - t_2, x_1 - x_2)\, \varphi(r_1, t_1, x_1)\, \overline{\psi(r_2, t_2, x_2)} \times r_1\, dr_1\, dt_1\, dx_1\, r_2\, dr_2\, dt_2\, dx_2 \tag{2.15}$$

with the integration intervals, $0 < r_j < \infty$, $0 \leqslant t_j \leqslant 2\pi$, $x_j \in \mathbb{R}$, $j = 1, 2$; and it follows that

$$(T(s)\varphi, T(s)\psi) = (\varphi, \psi), \quad \varphi, \psi \in \mathscr{D}, s \in \mathbb{R}.$$

It is clear that the two unitary groups $\{T(s): s \in \mathbb{R}\}$ and $\{R(\zeta): \zeta \in \mathbb{T}^1\}$ commute and, moreover, that both commute with L when L is viewed as an operator with dense domain $\mathscr{D}/\mathscr{K}$ in $\mathscr{H}_K$.

With the definition

$$Q(\varphi) = (L\varphi, \varphi), \qquad \varphi \in \mathscr{D}, \tag{2.16}$$

we have

$$Q(T(s)\varphi) = Q(R(\zeta)\varphi) = Q(\varphi), \qquad s \in \mathbb{R}, \zeta \in \mathbb{T}^1, \varphi \in \mathscr{D}.$$

Let $\mathscr{F}$ denote the completion of $\mathscr{D}/\mathscr{K}$ relative to the new quadratic form $\varphi \to Q(\varphi) + (\varphi, \varphi)$, and let $D(L^*)$ be the domain of the adjoint operator L^*, where the adjoint is defined relative to the original inner product $(\cdot, \cdot)$.

For the domain of the Friedrichs extension L_Q, we have (cf. [8]),

$$D(L_Q) = \mathscr{F} \cap D(L^*). \tag{2.17}$$

It follows that $D(L_Q)$ is invariant under both of the one-parameter groups, i.e., $T(s)\, D(L_Q) = R(\zeta)\, D(L_Q) = D(L_Q)$, and that $T(s)\, L_Q v = L_Q T(s) v$, and $R(\zeta)\, L_Q v = L_Q R(\zeta) v$ hold for $s \in \mathbb{R}$, $\zeta \in \mathbb{T}^1$, and $v \in D(L_Q)$.

The second conclusion in part (b) of the lemma follows from this. See [23, Chap. 1; 18, Chap. 3] for a more detailed discussion of commutation relations.

It remains to show that the infinitesimal generator of the unitary one-parameter group $\{T(s): s \in \mathbb{R}\}$ is the closure of its restriction to the dense subspace $\mathscr{D}/\mathscr{K}$ in $\mathscr{H}_K$. But this follows from the core-theorem [36, Corollary 1.3], since $\mathscr{D}/\mathscr{K}$ is invariant under $\{T(s): s \in \mathbb{R}\}$.

LEMMA 2.3. *Consider the representation* $\{R(\zeta) \exp(ipL_Q)\, T(s): \zeta \in \mathbb{T}^1, p, s \in \mathbb{R}\}$, *and let E be the spectral measure on the dual of* $\mathbb{T}^1 \times \mathbb{R}^2$, *i.e., on* $\mathbb{Z} \times \mathbb{R}^2$, *given by the* SNAG *theorem* [27] *through*

$$R(\zeta) \exp(ipL_Q)\, T(s) = \int_{\mathbb{Z} \times \mathbb{R}^2} \zeta^m e^{ip\lambda} e^{is\xi}\, dE(m, \lambda, \xi). \tag{2.18}$$

Then the projection valued measure E has multiplicity one on its support.

We take up the proof in the next section.

3. GENERALIZED EIGEN FUNCTIONS

In proving the multiplicity one result, we shall make use of direct integral theory as well as results from [22] on the sub-Laplacian.

Let Ω denote the support of the spectral measure E. We have $\Omega \subset \mathbb{Z} \times \mathbb{R}^2$, and the integral $\int_\Omega dE(\omega)$ represents the identity operator in the functional Hilbert space $\mathscr{H}_K$, defined by the given kernel function K on $B \times B$. Let U be the unitary representation from Lemma 2.2, given by

$$U(\zeta, p, s) = R(\zeta) \exp(ipL_Q)\, T(s). \tag{3.1}$$

Let

$$\mathscr{H}_K = \int^{\oplus} \mathscr{H}(\omega)\, d\omega \tag{3.2}$$

be the corresponding direct integral decomposition associated to the action of $U(\zeta, p, s)$. The Hilbert space $\int^{\oplus} \mathscr{H}(\omega)\, d\omega$ consists of measurable sections, relative to the spectral measure $d\omega$, and $\{\mathscr{H}(\omega)\}_{\omega \in \Omega}$ is a measurable family of Hilbert spaces. We refer to [30, 3] for details on direct integrals relative to a given abelian von Neumann algebra. We may identify vectors $v \in \mathscr{H}_K$ with measurable fields $\{v(\omega)\}_{\omega \in \Omega}$, $v(\omega) \in \mathscr{H}(\omega)$, satisfying

$$\|v\|^2_{\mathscr{H}_K} = \int_\Omega \|v(\omega)\|^2_\omega\, d\omega. \tag{3.3}$$

It follows from the spectral theorem (in its SNAG-form, see [27]) that

$$(U(\zeta, p, s)v)(\omega) = \zeta^m e^{ip\lambda} e^{is\xi} v(\omega) \tag{3.4}$$

when $\omega \in \Omega$ is given by coordinates $\omega = (m, \lambda, \xi) \in \mathbb{Z} \times \mathbb{R}^2$.

Let $C^\infty(U, \mathscr{H}_K)$ be the space of C^∞-vectors for the unitary representation U in $\mathscr{H}_K$, and let $\mathscr{H}_K^{-\infty}$ be the corresponding dual space. We refer to [11, 35, 36] for details on the spaces $C^\infty(U, \mathscr{H}_K)$ and $\mathscr{H}_K^{-\infty}$. We shall use here that elements in $\mathscr{H}_K^{-\infty}$ may be identified with measurable fields of distributions on B.

Let

$$J: \mathscr{D} \to \mathscr{D}/\mathscr{K} \to \mathscr{H}_K \tag{3.5}$$

be the composition of the quotent mapping from $\mathscr{D}$ to $\mathscr{D}/\mathscr{K}$, and the inclusion of $\mathscr{D}/\mathscr{K}$ into the corresponding Hilbert space completion $\mathscr{H}_K$. Since each map is continuous, J is continuous. Since $\mathscr{D}$ is a nuclear space when equipped with the inductive limit topology of L. Schwartz, it follows that there is a measurable field of continuous mappings

$$J_\omega: \mathscr{D} \to \mathscr{H}(\omega) \tag{3.6}$$

such that

$$J_\omega \varphi = (J\varphi)(\omega), \qquad \varphi \in \mathscr{D}, \omega \in \Omega.$$

In particular, each of the linear mappings J_ω: $\mathscr{D} \to \mathscr{H}(\omega)$, $\omega \in \Omega$ is continuous relative to the inductive limit topology on $\mathscr{D}$. This means that J_ω is a vector valued distribution. We shall write

$$J_\omega \varphi = \langle v(\omega), \varphi \rangle, \qquad \omega \in \Omega, \varphi \in \mathscr{D} \tag{3.7}$$

where it is understood that the distribution $v(\omega)$ is applied to the scalar-valued test function φ.

When formula (3.7) is applied to the vector $v = J\varphi \in \mathscr{H}_K$, we get, upon differentiating:

$$\left\langle v(\omega), \frac{\partial}{\partial t} \varphi \right\rangle = im \langle v(\omega), \varphi \rangle \tag{3.8}$$

$$\langle v(\omega), L\varphi \rangle = \lambda \langle v(\omega), \varphi \rangle \tag{3.9}$$

$$\left\langle v(\omega), \frac{\partial}{\partial x} \varphi \right\rangle = i\xi \langle v(\omega), \varphi \rangle \tag{3.10}$$

for $\omega = (m, \lambda, \xi) \in \Omega \subset \mathbb{Z} \times \mathbb{R}^2$, and $\varphi \in \mathscr{D} = C_c^\infty(B)$.

It follows, in particular, that the distribution $v(\omega)$ is a weak solution to the system of partial differential operators given by (3.8)–(3.10) and the three operators $\partial/\partial t$, L, and $\partial/\partial x$. But the operator L is known to be hypoelliptic [13], even analytic hypoelliptic [14]. It follows that the measurable field of distributions $\{v(\omega)\}_{\omega\in\Omega}$ from (3.7)–(3.10) is, in fact, a field of C^∞-functions on B. Moreover, (3.12)–(3.14) below are satisfied in the strong sense. Specifically, let $v(\omega, r, t, x)$ be the C^∞-function on B which defines the distribution $v(\omega)$. We have

$$\langle v(\omega), \varphi\rangle = \int_{\mathbb{R}} \int_0^{2\pi} \int_{\mathbb{R}_+} v(\omega, r, t, x)\, \varphi(r, t, x) r\, dr\, dt\, dx \tag{3.11}$$

and, from (3.8)–(3.10), we obtain

$$\frac{\partial}{\partial t} v(\omega, r, t, x) = -imv(\omega, r, t, x), \tag{3.12}$$

$$Lv(\omega, r, t, x) = \lambda v(\omega, r, t, x), \tag{3.13}$$

$$\frac{\partial}{\partial x} v(\omega, r, t, x) = -i\xi v(\omega, r, t, x), \tag{3.14}$$

for $\omega = (m, \lambda, \xi) \in \Omega$, and $g = (r, t, x) \in B$.

It follows that the last two variables t, x may be separated, and

$$v(\omega, r, x, t) = f(\omega, r)\, e^{-imt} e^{-i\xi x},$$

where the function $r \to f(\omega, r)$ is C^∞ on $[0, 1)$. Since the function $(z, x) \to v(\omega, z, x)$ is C^∞ in $\{z \in \mathbb{C}\colon |z| < 1\} \times \mathbb{R}$, it follows that $r \to f(\omega, r)$ is C^∞ also at $r = 0$, and, in particular, continuous at $r = 0$. But

$$L = -\frac{1}{4}\left(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{\partial}{\partial t}\right)^2\right) - r^2\left(\frac{\partial}{\partial x}\right)^2 + \frac{\partial^2}{\partial t\, \partial x}, \tag{3.15}$$

and it follows that the function $f(\omega, \cdot)$ satisfies

$$\left(-\frac{1}{4}\left(\frac{1}{r}\frac{d}{dr} r \frac{d}{dr} - \frac{m^2}{r^2}\right) + |\xi|^2 r^2 - m\xi\right) f(\omega, r) = \lambda f(\omega, r). \tag{3.16}$$

Let (m, λ, ξ) be such that the equation

$$\lambda = (2j + 1)\, |\xi| + 2m\xi$$

has solutions $j \in 2^{-1}\mathbb{Z}$, $j \geqslant 0$. In [22] we found normalized solutions to the system (3.12)–(3.14). Let h_ω be the function on $0 < r < \infty$ which is defined by

$$h_\omega(r) = \sqrt{\frac{(j-|m|)!\, 2^{2|m|+1}}{\Gamma(j+|m|+1)}}\,(|\xi|\, r^2)^{|m|} \times L_{j-|m|}^{2|m|}(2\,|\xi|\, r^2) e^{-|\xi|\, r^2}, \tag{3.17}$$

where $L^{\cdot}_{\cdot}$ denotes the familiar Laguerre functions. Then $h_\omega(r)$ satisfies (3.16), and

$$\int_0^\infty |h_\omega(r)|^2\, r\, dr = 1$$

when $j,\ m \in 2^{-1}\mathbb{Z}$, $j \geqslant 0$, $|m| \leqslant j$, and

$$\lambda = (2j+1)\,|\xi| + 2m\xi. \tag{3.18}$$

There is a second independent solution $\tilde{h}_\omega(r)$ to (3.16), but it satisfies $\lim_{r \to 0} \tilde{h}_\omega(r) = \infty$. As mentioned, we must exclude this second singular solution by virtue of the hypoellipticity of the sub-Laplacian L.

Also, note that the half-integral values of j (and m) must be excluded since they do not define functions on the Heisenberg group G, but only on a twofold cover of G.

We conclude that there are measurable functions $A(j, m, \xi)$ such that

$$f(\omega, r) = A(j, m, \xi)\, h(j, m, \xi, r) \tag{3.19}$$

and

$$v(\omega, r, t, x) = A(j, m, \xi)\, h(j, m, \xi, r) e^{-i(mt + \xi x)}. \tag{3.20}$$

The point $\omega = (m, \lambda, \xi)$ is in the support Ω of the spectral measure, if and only if, $\lambda \geqslant 0$ and $\lambda \in (2\mathbb{Z}+1)\xi$. In that case, there is a unique $j \in \mathbb{Z}$, $j \geqslant 0$ satisfying (3.18). We thus have an identification $(m, \lambda, \xi) \to (j, m, \xi)$ with j determined by (3.18) for $(m, \lambda, \xi) \in \Omega$.

This concludes the proof that the spectral multiplicity is one in the support of the spectral measure E.

4. An Explicit Formula for the Integral Kernel

We now show how the extension problem for the initially given kernel function $K(g_1, g_2)$ on $B \times B$ can be solved by means of the eigenfunction analysis in Section 3.

We are ready to prove Theorem 1.1.

Proof (Theorem 1.1). Let K be the integral kernel on $B \times B$ satisfying the invariance conditions, and the positivity, in the statement of Theorem 1.1. Let $\mathscr{H}_K$ be the Hilbert space constructed from K by the generalized GNS-construction, and let J: $\mathscr{D} \to \mathscr{D}/\mathscr{K} \to \mathscr{H}_K$ be defined as in the proof of Lemma 2.3 above. Let $v(\omega)$ be the corresponding measurable field of C^∞-functions determined by the system (3.12)–(3.15) of differential equations. We have

$$J\varphi(\omega) = J_\omega \varphi = \langle v(\omega), \varphi \rangle = \int_B v(\omega, g)\, \varphi(g)\, dg,$$

where dg is the Haar measure on G.

We also have the explicit formulas (3.17)–(3.20) for the family of functions, $g \to v(\omega, g)$, and we note that each function extends naturally from B to all of G since the explicit formula is automatically defined everywhere in G.

Since

$$(\varphi, \psi) = \int_B \int_B K(g_1, g_2)\, \varphi(g_1)\, \overline{\psi(g_2)}\, dg_1\, dg_2$$

and

$$(\varphi, \psi) = \int_\Omega \langle v(\omega), \varphi \rangle\, \overline{\langle v(\omega), \psi \rangle}\, d\omega,$$

an application of Fubini's theorem yields

$$K(g_1, g_2) = \int_\Omega v(\omega, g_1)\, \overline{v(\omega, g_2)}\, d\omega. \tag{4.1}$$

But

$$v(\omega, r, t, x) = A(j, m, \xi)\, h(j, m, \xi, r) e^{-i(mt + \xi x)}. \tag{4.2}$$

If we define a measure μ in

$$\{(j, m, \xi) \colon j, m \in \mathbb{Z},\ j \geqslant 0,\ |m| \leqslant j,\ \xi \in \mathbb{R}\} \tag{4.3}$$

by

$$d\mu(j, m, \xi) = |A(j, m, \xi)|^2\, d\omega, \tag{4.4}$$

it follows that

$$K(r_1, r_2, t, x) = \int_{\mathbb{R}} \sum_j \sum_m h(j, m, \xi, r_1)\, h(j, m, \xi, r_2) e^{-i(mt + \xi x)}\, d\mu(j, m, \xi). \tag{4.5}$$

This formula is valid in the region $0 < r_1 < 1$, $0 < r_2 < 1$, $t \in \mathbb{R}/2\pi\mathbb{Z}$, $x \in \mathbb{R}$. In terms of the original coordinates, $g_k = (r_k, t_k, x_k)$, $k = 1, 2$, $K(g_1, g_2) = K(r_1, r_2, t_1 - t_2, x_1 - x_2)$ with the right-hand side of this expression given by formula (4.5) above.

We prove parts (b) and (c) in Theorem 1.1 by the same argument. We consider functions K on $B \times B$ which admit an integral representation of the form

$$K(g_1, g_2) = \int_{\Omega} v(\omega, g_1)\, \overline{v(\omega, g_2)}\, d\mu(\omega)$$

for some finite Radon measure μ on Ω and a measurable field $\{v(\omega, \cdot)\colon \omega \in \Omega\}$. We proved in (a) that the functions $v(\omega, \cdot)$ may be taken to be C^∞ on B with C^∞ extensions to G.

For $\varphi \in \mathscr{D}$, we have the following computation, valid by virtue of Fubini's theorem:

$$\begin{aligned}
&\int_B \int_B K(g_1, g_2)\, \varphi(g_1)\, \overline{\varphi(g_2)}\, dg_1\, dg_2 \\
&\quad = \int_{\Omega} \int_B v(\omega, g_1)\, \varphi(g_1)\, dg_1 \int_B \overline{v(\omega, g_2)}\, \overline{\varphi(g_2)}\, dg_2\, d\mu(\omega) \\
&\quad = \int_{\Omega} \left| \int_B v(\omega, g)\, \varphi(g)\, dg \right|^2 d\mu(\omega) \geqslant 0.
\end{aligned}$$

This estimate is also valid when the integration is extended over $G \times G$, rather than just $B \times B$, for functions $\varphi \in C_c^\infty(G)$.

This completes the proof that the kernel is positive definite, also when extended to $G \times G$, for arbitrary positive, finite Radon measures μ.

5. Positive Definite Functions in a Single Variable

Let G be the Heisenberg group, and let B be an open subset of G. Let the subset C be defined by

$$C = BB^{-1} = \{g_1 g_2^{-1}\colon g_1, g_2 \in B\}. \tag{5.1}$$

A function F defined on C is said to be *positive definite* if

$$\sum_i \sum_j F(g_i \cdot g_j^{-1}) \zeta_i \bar{\zeta}_j \geqslant 0 \tag{5.2}$$

for all $n = 1, 2, ..., g_1, ..., g_n \in B$, $\zeta_1, ..., \zeta_n \in \mathbb{C}$. If F is also assumed continuous, this is equivalent to the integrated form of the condition:

$$\int_B \int_B F(g_1 \cdot g_2^{-1}) \varphi(g_1) \overline{\varphi(g_2)}\, dg_1\, dg_2 \geqslant 0 \qquad \text{for all} \quad \varphi \in C_c^\infty(B),$$

and it is also equivalent to the statement that the kernel function K, defined on $B \times B$ by

$$K(g_1, g_2) = F(g_1 \cdot g_2^{-1}), \tag{5.3}$$

be positive definite, as defined by (1.4) in Section 1.

If B is the unit cylinder $B = \{(z, x) \in G: |z| < 1\}$, then the set C, defined in (5.1), is also a cylinder, $C = 2B = \{(z, x) \in G: |z| < 2\}$.

We shall prove the following result on continuous positive definite functions on $2B$.

COROLLARY 5.1. *Let F be a continuous positive definite function defined in the cylinder $2B$ in the Heisenberg group G, and assume that F is invariant under the $\mathbb{T}^1$-action, i.e.,*

$$F(\zeta z, x) = F(z, x), \qquad \zeta \in \mathbb{T}^1, (z, x) \in 2B.$$

Let $\mathbb{Z}_+ = \{0, 1, 2, ...\}$. Then there is a finite Radon measure on $\mathbb{Z}_+ \times \mathbb{R}$ such that

$$F(r, t, x) = \int_{\mathbb{R}} \sum_j L_j(2\,|\xi|\, r^2)\, e^{-|\xi| r^2} e^{-i\xi x} \mu(j, d\xi) \tag{5.4}$$

for all points (r, t, x) in B, i.e., r satisfying $0 \leqslant r < 1$. (We denote the Laguerre polynomial L_j^0 by L_j for simplicity of notation.)

Proof. The result follows from Theorem 1.1 once we verify that the kernel K, defined by (5.4) above, satisfies the conditions from the theorem. We must verify that K is bi-invariant and strongly positive definite as a function (in two variables) on $B \times B$.

For this purpose, it is useful to introduce the Lie algebra $\mathfrak{g}$ of the Heisenberg group, and the corresponding complex universal enveloping algebra

$\mathfrak{A}$. Recall that $\mathfrak{A}$ is a $*$-algebra when the $*$-operation on $\mathfrak{A}$ is defined by extension from $\mathfrak{g}$ with

$$X^* = -X, \qquad X \in \mathfrak{g}.$$

Let ρ be the right-regular representation given by

$$\rho(X)\,\varphi(g) = \frac{d}{dt}\bigg|_{t=0} \varphi(g \cdot \exp tX). \tag{5.5}$$

We have the following:

LEMMA 5.2. *Let $\mathscr{H}_F$ be the Hilbert space obtained by applying the GNS-construction to the quadratic form (sesquilinear),*

$$(\varphi, \psi) = \int_B \int_B F(g_1 \cdot g_2^{-1})\,\varphi(g_1)\,\overline{\psi(g_2)}\,dg_1\,dg_2, \tag{5.6}$$

defined for φ, $\psi \in C_c^\infty(B)$. Then the representation ρ passes to the quotient and defines a $*$-representation of the enveloping algebra $\mathfrak{A}$.

Proof. The result follows from familiar properties of the convolution algebra $C_c^\infty(G)$ which contains $\mathscr{D} = C_c^\infty(B)$ via the natural inclusion. We define the convolution $\times$ by

$$\varphi \times \psi(g) = \int_G \varphi(g \cdot h^{-1})\,\psi(h)\,dh \tag{5.7}$$

and the $*$-operation

$$\psi^*(g) = \overline{\psi(g^{-1})}, \qquad g \in G, \tag{5.8}$$

for functions φ, ψ in $C_c^\infty(G)$. When φ and ψ are restricted to $\mathscr{D} = C_c^\infty(B)$, we have the formula

$$(\varphi, \psi) = \int_{2B} F(g)(\varphi \times \psi^*)(g)\,dg. \tag{5.9}$$

The verification of this formula, for the inner product from (5.6), follows from Fubini's theorem and the invariance (to the right) of Haar measure.

It now follows immediately that

$$(\rho(X)\varphi, \psi) = -(\varphi, \rho(X)\psi)$$

holds for $X \in \mathfrak{g}$, and $\varphi, \psi \in \mathscr{D}$. More generally, we get

$$(\rho(A)\varphi, \psi) = (\varphi, \rho(A^*)\psi) \qquad \text{for} \quad A \in \mathfrak{A}, \varphi, \psi \in \mathscr{D}. \tag{5.10}$$

We have, in particular, $(\rho(Z)\varphi, \psi) = -(\varphi, \rho(\bar{Z})\psi)$, where the two elements Z and $\bar{Z}$ in the complexified Lie algebra are given by formulas (2.6)–(2.7). By (2.8), it follows that

$$(\rho(L)\varphi, \psi) = (\varphi, \rho(L)\psi)$$

and

$$(\rho(L)\varphi, \varphi) = \tfrac{1}{2}(\|\rho(Z)\varphi\|^2 + \|\rho(\bar{Z})\varphi\|^2) \qquad \text{for all} \quad \varphi, \psi \in \mathscr{D}.$$

We therefore have $(\rho(L)\varphi, \varphi) \geqslant 0$ which means that the kernel K, defined by (4.5) from the given function F, is indeed strongly positive definite.

It is immediate that this kernel is also bi-invariant since the vector field $\partial/\partial x$ is in the Lie algebra $\rho(\mathfrak{g})$ of operators on $\mathscr{H}_F$.

This concludes the proof of Corollary 5.1, since Theorem 1.1 has been shown to apply. The desired formula (5.4) follows from the observation that $h(\omega, 0) = 0$ if $\omega = (j, m, \xi)$ satisfies $m \neq 0$. This is because the function $h(j, m, \xi, r)$ contains the factor $(|\xi| \, r^2)^{|m|}$.

THEOREM 5.3. *Let F be a continuous positive function in the cylinder $2B$ of the Heisenberg group G, and let K be the kernel function on $G \times G$ constructed in Theorem 1.1 and Corollary 5.1 and satisfying*

$$F(g_1 \cdot g_2^{-1}) = K(g_1, g_2) \qquad \text{in} \quad B \times B.$$

Suppose the function K is analytic. Then it follows that there is a positive definite function $\tilde{F}$ on G such that

$$\tilde{F}(g_1 \cdot g_2^{-1}) = K(g_1, g_2) \qquad \text{in} \quad G \times G. \tag{5.11}$$

In particular, the function $\tilde{F}$ extends F on $2B$.

Proof. Let the function k be defined on $B \times B$ by

$$k(g_1, g_2) = F(g_1 \cdot g_2^{-1}).$$

We noted that Theorem 1.1 applies to this function and yields an extension K, defined on $G \times G$; and we are assuming that K is analytic.

For $X \in \mathfrak{g}$, the Lie algebra of G, define vector fields on $G \times G$ by the formula

$$\rho(X)\, K(g_1, g_2) = \frac{d}{dt}\bigg|_{t=0} K(g_1 \cdot \exp tX, g_2 \cdot \exp tX). \tag{5.12}$$

Let $(g_1, g_2) \in B \times B$, and let $a \in G$ be chosen such that $(g_1 \cdot a, g_2 \cdot a) \in B \times B$. We then have

$$K(g_1 \cdot a, g_2 \cdot a) = F(g_1 \cdot a \cdot (g_2 \cdot a)^{-1}) = F(g_1 \cdot a \cdot a^{-1} \cdot g^{-1})$$
$$= F(g_1 \cdot g_2^{-1}) = K(g_1, g_2).$$

By taking $a = \exp tX$, we note that $\rho(X)K \equiv 0$ in $B \times B$. But K, and its derivative, $\rho(X)K$, are analytic on $G \times G$. It follows that $\rho(X)K$ vanishes on $G \times G$ for all X in the Lie algebra. This implies that

$$K(g_1 \cdot a, g_2 \cdot a) = K(g_1, g_2) \qquad \text{for all points} \quad a, g_1, g_2 \text{ in } G. \tag{5.13}$$

First consider a of the form $a = \exp X$, and note that

$$K(g_1 \cdot a, g_2 \cdot a) - K(g_1, g_2) = \int_0^1 \rho(X)\, K(g_1 \cdot \exp tX, g_2 \cdot \exp tX)\, dt = 0.$$

Since G is connected, the desired conclusion (5.14) then holds for all a in G. Alternatively, use that the exponential mapping is onto for the Heisenberg group.

We define the desired function $\tilde{F}$ on G by $\tilde{F}(g) = K(g, e)$ and note that

$$\tilde{F}(g_1 \cdot g_2^{-1}) = K(g_1 \cdot g_2^{-1}, e) = K(g_1 \cdot g_2^{-1} \cdot g_2, g_2) = K(g_1, g_2).$$

Therefore,

$$\int_G \int_G \tilde{F}(g_1 \cdot g_2^{-1})\, \varphi(g_1)\, \overline{\varphi(g_2)}\, dg_1\, dg_2$$
$$= \int_G \int_G K(g_1, g_2)\, \varphi(g_1)\, \overline{\varphi(g_2)}\, dg_1\, dg_2 \geqslant 0$$

by Theorem 1.1. This concludes the proof that $\tilde{F}$ is a positive definite extension.

Strictly speaking, we have only checked

$$\tilde{F}(g) = F(g) \qquad \text{for} \quad g \in B, \tag{5.14}$$

and the given function F was defined on the larger cone $2B$. But (5.14) must hold on the larger set as well by the uniqueness of analytic extensions.

6. THE HEAT KERNEL AND THE FRIEDRICHS EXTENSION

EXAMPLE 6.1. Let $\{P(s, g): s \in \mathbb{R}, g \in G\}$ be the convolution kernel on the Heisenberg group which solves the heat equation

$$\left(\frac{\partial}{\partial s} + L\right) P(s, g) = \delta(s, g) \qquad \text{(Dirac delta)}$$

$$P(s, g) = 0, \qquad s < 0.$$

Then the function $g \to P(s, g)$, satisfies the conditions in Corollary 5.1 for all $s \in \mathbb{R}_+$.

Proof. In [22] we found the formula below for the heat kernel. Let $P_s(g) = P(s, g)$. Then we have, in polar coordinates, $g = (r, t, x) \in \mathbb{R}_+ \times (\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}$:

$$\begin{aligned} P_s(r, t, x) &= \frac{1}{2\pi^2} \int_{\mathbb{R}} \sum_j \frac{1}{\sqrt{(2j)!}} L_j(2\,|\xi|\, r^2)\, e^{-s(2j+1)|\xi|} e^{-|\xi| r^2} \cos(\xi x)\, d\xi \\ &= \frac{1}{\pi^2} \int_0^\infty \sum_j \frac{1}{\sqrt{(2j)!}} L_j(2\xi r^2)\, e^{-s(2j+1)\xi}\, e^{-\xi r^2} \cos(\xi x)\, d\xi. \end{aligned} \tag{6.1}$$

Since $g^{-1} = (r, t+\pi, -x)$ if $g = (r, t, x)$, it follows that $P_s(g^{-1}) = P_s(g)$. We have a self-adjoint semigroup of positive operators,

$$H_s \varphi(g) = \int_G P_s(h)\, \varphi(h \cdot g)\, dh \tag{6.2}$$

with

$$\begin{aligned} (H_s \varphi, \varphi)_{\mathscr{L}^2} &= \int_G H_s \varphi(g)\, \overline{\varphi(g)}\, dg \\ &= \int_G \int_G P_s(h \cdot g^{-1})\, \varphi(h)\, \overline{\varphi(g)}\, dh\, dg > 0. \end{aligned} \tag{6.3}$$

This concludes the proof of the assertion in Example 6.1.

The proof of Theorem 1.1 uses the Friedrichs extension L_Q which, we recall, is a self-adjoint operator in the Hilbert space $\mathscr{H}$ associated to the given positive definite function F, defined in the open set $C = B \cdot B^{-1}$ as specified above. It is obtained by applying the GNS-construction to $C_c^\infty(B)$, relative to the inner product, $(\varphi, \psi) = F(\varphi \times \psi^*)$.

We conclude this section by noting that the Friedrichs extension may be expressed in terms of the heat kernel $\{P_s : s \in \mathbb{R}_+\}$ as follows: We recall that the heat kernel $\{P_s\}$ determines a Wiener measure on the path space in G. The construction of this measure E_0 on the set of paths, $t \to \gamma(t)$ in G, is done in [29, 44].

Let $\Gamma(B, s)$ be the set of paths:

$$\Gamma(B, s) = \{\gamma : \gamma(t) \in B, 0 \leqslant t \leqslant s\},$$

and let $(\cdot, \cdot)$ denote the inner product of the Hilbert space $\mathscr{H}$. For the semi-group, $\exp(-sL_Q)$ generated by the self-adjoint Friedrichs operator L_Q, we have

$$(\exp(-sL_Q)\varphi, \psi) = \int_{\Gamma(B,s)} \varphi(\gamma(s))\, \psi(\gamma(0))\, dE_0(\gamma). \tag{6.4}$$

The proof of this formula is given, in a special case, in [44, Theorem 2.1.1, p. 224], and the ideas from the special case carry over to the present setting. We shall omit details since the result will not be used in the sequel.

7. Extensions of Representations and of Functions

Let G be a Lie group with Lie algebra $\mathfrak{g}$, and let $\mathfrak{U}$ denote the complex universal enveloping algebra of $\mathfrak{g}$. Let dg denote a right-invariant Haar measure on G. Let U be a unitary representation (strongly continuous) of G on a Hilbert space $\mathscr{H}$, and let dU be the corresponding derived representation of $\mathfrak{A}$. For elements $X \in \mathfrak{g}$, and for vectors v in the Gårding space, this representation is given by

$$dU(X)v = \frac{d}{dt}\bigg|_{t=0} U(\exp tX)v. \tag{7.1}$$

We refer to [23, 18] for more details on this representation, and the definition of the Gårding space. We note that dU is hermitian, i.e., that

$$(dU(A)v_1, v_2) = (v_1, dU(A^*)v_2) \tag{7.2}$$

holds for all $A \in \mathfrak{A}$, and for all Gårding vectors v_1, v_2. We also note that *not* every hermitian representation ρ, with a dense domain in a Hilbert space, is of the form dU for some unitary representation U of a Lie group with $\mathfrak{g}$ as its Lie algebra. We say that ρ is *exact* (or *integrable*) if it is of this form, i.e., $\rho = dU$. We shall say that a hermitian representation ρ of $\mathfrak{A}$, with dense domain $D(\rho)$ in a Hilbert space $\mathscr{H}$, is *extendable* if there is a Hilbert space $\tilde{\mathscr{H}}$, an isometry S: $\mathscr{H} \to \tilde{\mathscr{H}}$, and a unitary representation U

of G (or some covering group for G) on $\tilde{\mathscr{H}}$ with Gårding space $\tilde{\mathscr{H}}(U)$ such that:

(i) $S(D(\rho)) \subset \tilde{\mathscr{H}}(U)$, and

(ii) $dU(A)\, Sv = S\rho(A)v$

hold for all $A \in \mathfrak{A}$ and $v \in D(\rho)$.

(We note that a slightly weaker version of (i) is frequently convenient: The Gårding space $\tilde{\mathscr{H}}(U)$ coincides with the space of C^∞-vectors for U, cf. [7], and it is a Fréchet space in the C^∞-topology defined by dU. Instead of (i), it suffices to assume

(i*) The space $S(D(\rho))$ is dense in a closed subspace of $\tilde{\mathscr{H}}(U)$.

Let B be a neighborhood of the origin e in G, and let

$$C = BB^{-1} = \{ g_1 \cdot g_2^{-1} \colon g_1, g_2 \in B \}. \tag{7.3}$$

A positive definite continuous function F on C is said to be extendable if there is a positive definite function $\tilde{F}$, defined on G, such that $\tilde{F} \equiv F$ on C. We say that F is extendable in the sense of distributions if a positive definite distribution $\tilde{F}$ exists, extending F.

We say that the extension $(U, \tilde{\mathscr{H}})$ is *minimal* if the vectors $\{U(g)\, S\varphi\colon g \in G, \varphi \in \mathscr{H}\}$ generate the whole space $\tilde{\mathscr{H}}$, i.e., the *closed* span is the whole space.

If the representation ρ, with dense domain $D(\rho)$ in $\mathscr{H}$, has an extension, then it also has a minimal extension; for let $(U, \tilde{\mathscr{H}})$ be a given extension with isometry $S\colon \mathscr{H} \to \tilde{\mathscr{H}}$, satisfying conditions (i)–(ii), and let $\tilde{\mathscr{H}}_0$ be the closed span of $\{U(g)\, S\varphi\colon g \in G,\ \varphi \in \mathscr{H}\}$. Since the subspace $\tilde{\mathscr{H}}_0 \subset \tilde{\mathscr{H}}$ is clearly invariant under U, it follows that the restricted representation $(U|_{\tilde{\mathscr{H}}_0}, \tilde{\mathscr{H}}_0)$ is also an extension.

THEOREM 7.1. *Let B be an open connected set in G containing e, and let $C = BB^{-1}$. Let F be a continuous positive definite function defined on C. Let $\mathscr{H}$ be the Hilbert space obtained from $C_c^\infty(B)$, and the inner product*

$$(\varphi, \psi) := \int_B \int_B F(g_1 \cdot g_2^{-1})\, \varphi(g_1)\, \overline{\psi(g_2)}\, dg_1\, dg_2, \tag{7.4}$$

where we pass to the quotient by $\{\varphi\colon (\varphi, \varphi) = 0\}$, and to the norm completion of this quotient.

Let ρ denote the hermitian representation of $\mathfrak{A}$ on $\mathscr{H}$ which is given by

$$\rho(X)\, \varphi(g) = \left.\frac{d}{dt}\right|_{t=0} \varphi(g \cdot \exp tX), \qquad \varphi \in C_c^\infty(B),\ X \in \mathfrak{g},\ g \in B. \tag{7.5}$$

If the given function F is extendable, then it follows that the representation ρ is extendable to an integrable representation in a possibly larger Hilbert space.

Remark 7.2. It is not known whether or not the converse implication holds for arbitrary Lie groups G, but it does in a weakened sense if $G = \mathbb{R}^n$, or if G is the Heisenberg group.

Specifically,

THEOREM 7.3. *Let G be an abelian Lie group or the Heisenberg group. Let B be an open connected set containing e, and let $C = BB^{-1}$. Let F be a continuous positive definite function on C, and let ρ be the corresponding representation of the enveloping algebra described in Theorem* 5.3 *above.*

If the representation ρ is extendable, then it follows that the given function F is extendable in the sense of distributions.

Remark 7.4. There is a rich literature on extendable hermitian representations (also called $*$-representations), and we note some of the references: [37, 10, 41, 17–19, 2, 24].

It follows from Rudin's example [38] (a nonextendable positive definite function on the n-dimensional cube I^n in $\mathbb{R}^n$, $n \geqslant 2$) that hermitian representations are generally not extendable.

A similar notion of extendability may be defined for distributions, and it is related to an extendability question for a certain representation studied by Fuglede [9], Jorgensen [19], and Pedersen [34].

THEOREM 7.5. *Let G be an abelian Lie group, or the Heisenberg group. Let B be a connected open subset containing e, and let F be a positive definite distribution on $B \cdot B^{-1}$, i.e., $F(\varphi \times \varphi^*) \geqslant 0$ holds for all $\varphi \in C_c^\infty(B)$. Assume that the corresponding representation ρ of the Lie algebra is extendable. Then it follows that F has an extension to G, i.e., there is a positive definite distribution $\tilde{F}$ on G such that $\tilde{F} = F$ on $B \cdot B^{-1}$.*

In the case when G is abelian, we have the stronger result:

PROPOSITION 7.6. (a) *Let G be an abelian Lie group, and let F be a continuous positive definite function as specified in Theorem* 5.3. *Then the enveloping algebra $\mathfrak{A}$ may be identified with the ring of all polynomials on $\hat{G}$. Let $\mathfrak{A}_+$ denote the cone given by*

$$\mathfrak{A}_+ := \{P \in \mathfrak{A}\colon P(\xi) \geqslant 0,\ \xi \in \hat{G}\}. \tag{7.6}$$

Let ρ denote the infinitesimal representation of $\mathfrak{A}$. Then ρ is extendable, if and only if $(\rho(P)\varphi, \varphi) \geqslant 0$ for all $\varphi \in C_c^\infty(B)$ and all $P \in \mathfrak{A}_+$.

(b) *The possible minimal extensions of* ρ *on* $\mathscr{H}$ *are parametrized by the set of finite positive Radon measures on* $\hat{G}$. *For such a measure* μ, *the extension may be chosen as follows*: $(U, \mathscr{L}^2(G, \mu), S_\mu)$ *with*

(i) $U(a) f(\xi) = \langle a, \xi \rangle f(\xi)$, $a \in G$, $\xi \in \hat{G}$, $f \in \mathscr{L}^2(\hat{G}, \mu)$,

(ii) $S_\mu \varphi(\xi) = \hat{\varphi}(\xi)$, $S_\mu \colon \mathscr{H} \to \mathscr{L}^2(\hat{G}, \mu)$, *and*

(iii) $dU(X) S_\mu \varphi(\xi) = i\langle X, \xi \rangle \hat{\varphi}(\xi) = S_\mu \rho(X) \varphi(\xi)$, *with* $\langle X, \xi \rangle = \sum_j x_j \xi_j$.

(c) *Let* $\mathscr{A}(B)$ *be the algebra of trigonometric polynomials on* $\hat{G}$ *which is spanned by the functions*:

$$\xi \to \langle t, \xi \rangle = \xi(t), \qquad t \in B. \tag{7.7}$$

Then it is possible to choose an extension within *the Hilbert space* $\mathscr{H}$, *if and only if the measure* μ *may be chosen so that* $\mathscr{A}(B)$ *is dense in* $\mathscr{L}^2(\hat{G}, \mu)$.

The extendability question for representations ρ, associated to the group $G = \mathbb{R}^2$, is also related to the question of finding normal extensions (or dilations) to formally normal operators in a given Hilbert space, when the extension is considered in a possibly larger Hilbert space.

It is classical [4] that such extensions are not always possible, and counterexamples are explored more recently by Schmüdgen in the theory of algebras of unbounded operators [41].

Proof of Theorem 7.1. A normalization constant is chosen once and for all, and we denote the corresponding modular function by Δ. It satisfies

$$\int_G \varphi(g^{-1}) \Delta(g^{-1}) \, dg = \int_G \varphi(g) \, dg \tag{7.8}$$

for all $\varphi \in \mathscr{D}(G) := C_c^\infty(G)$.

We have a given continuous positive definite function F defined on $C = BB^{-1}$, and it is assumed that F is extendable. Let $\tilde{F}$ denote a positive definite extension to all of G. Then $\tilde{F}$ is known to be continuous.

When the Gelfand–Naimark–Segal (GNS) construction [12] is applied to $\tilde{F}$, we obtain a cyclic representation $(U, \tilde{\mathscr{H}}, v_0)$, where U is a strongly continuous representation of G in the Hilbert space $\tilde{\mathscr{H}}$ and v_0 is a cyclic vector, $v_0 \in \tilde{\mathscr{H}}$, for this representation. We have

$$\tilde{F}(g) = (v_0, U(g) v_0), \qquad g \in G. \tag{7.9}$$

Moreover, the cyclic representation $(U, \tilde{\mathscr{H}}, v_0)$ is determined, up to unitary equivalence, by the function F.

One way of obtaining the Hilbert space $\tilde{\mathscr{H}}$ is as follows: Set

$$(\varphi, \psi) := \int_G \tilde{F}(g)(\varphi \times \psi^*)(g)\, dg,$$

where $\times$ denotes the convolution, and

$$\psi^*(g) = \Delta(f^{-1})\, \overline{\psi(g^{-1})}, \qquad g \in G,$$

φ, $\psi \in \mathscr{D}(G)$. Define $\tilde{F}(\varphi) := \int_G \tilde{F}(g)\, \varphi(g)\, dg$ and $\tilde{F}(\delta) := \tilde{F}(e)$, where δ is the Dirac delta function. Then $\tilde{F}$ is a positive linear functional on the unital algebra $\mathscr{D}(G) \oplus \mathbb{C}\, \delta$, obtained from the convolution algebra $\mathscr{D}(G)$ by adjoining the unit element δ.

Now, pass to the quotient by $(\varphi, \varphi) = 0$ and the corresponding Hilbert space completion $\tilde{\mathscr{H}}$. We have a natural inclusion of the quotient as a dense subspace in $\tilde{\mathscr{H}}$, and the vector v_0 will be the image of δ under this inclusion. With these choices, formula (7.9) can be verified.

If $\varphi \in C_c^\infty(B) \subset \mathscr{D}(G)$, we define

$$S(\varphi) = U(\varphi) v_0, \tag{7.10}$$

where

$$U(\varphi) := \int_G \varphi(g)\, U(g^{-1})\, dg = \int_G \Delta(g^{-1})\, \varphi(g^{-1})\, U(g)\, dg. \tag{7.11}$$

The inner product on $C_c^\infty(B)$ is easily computed to be

$$(S(\varphi), S(\psi)) = \int_B \int_B F(g_1 \cdot g_2^{-1})\, \varphi(g_1)\, \overline{\psi(g_2)}\, dg_1\, dg_2$$

and note that S passes to the quotient in $C_c^\infty(B)$ by $\{\varphi\colon (\varphi, \varphi) = 0\}$ and further extends, by closure, to an isometry, $S\colon \mathscr{H} \to \tilde{\mathscr{H}}$.

The Hilbert space $\mathscr{H}$ contains a dense subspace $D(\rho)$, obtained by passing $C_c^\infty(B)$ to the quotient. If $\varphi \in C_c^\infty(B)$, it shall be convenient do denote the corresponding element in $D(\rho) \subset \mathscr{H}$ also by φ. For the representation ρ, we have

$$\rho(X)\, \varphi(g) = \frac{d}{dt}\bigg|_{t=0} \varphi(g \cdot \exp tX) \qquad \text{for} \quad X \in \mathfrak{g}, \text{ and } g \in B. \tag{7.12}$$

We claim that the representation dU, with the Gårding space $\tilde{\mathscr{H}}(U)$ as domain, extends the given representation ρ on the domain $D(\rho)$.

Specifically, that

$$S(\rho(X)\varphi) = dU(X)\, S(\varphi), \qquad X \in \mathfrak{g},\ \varphi \in D(\rho), \tag{7.13}$$

as is easily computed.

Proof of Theorem 7.3. *Case* 1. We assume G is an abelian Lie group, deferring the proof for the Heisenberg group until after Corollary 7.8. The analysis may easily be reduced to the case $G = \mathbb{R}^n \times \mathbb{T}^m$, so that the dual group $\hat{G}$ of unitary characters is $G \simeq \mathbb{R}^n \times \mathbb{Z}^m$. We shall denote the pairing between $t \in G$ and $\xi \in \hat{G}$ by $\langle t, \xi \rangle = \xi(t)$. If $t = (t_1, ..., t_n, \zeta_1, ..., \zeta_m)$ with $t_i \in \mathbb{R}$, $\zeta_j \in \mathbb{T}^1$, and $\xi = (\xi_1, ..., \xi_n, k_1, ..., k_m)$ with $\xi_i \in \mathbb{R}$, $k_j \in \mathbb{Z}$, then

$$\langle t, \xi \rangle = \xi(t) = \prod_i e^{\sqrt{-1}\, t_i \xi_i} \prod_j \zeta_j^{k_j}. \tag{7.14}$$

For a given positive definite function F, defined on the given set $C = B \cdot B^{-1}$ (or $B - B$ if the operation is written additively), we construct a Hilbert space $\mathscr{H}$ as in the proof of Theorem 7.1.

The Lie algebra of G is just $\mathbb{R}^{n+m} = \mathfrak{g}$, and the operator $\rho(X)$, $X \in \mathfrak{g}$, is a vector field with constant coefficients. Choose coordinates $(t_1, ..., t_{n+m})$ on G with $t_j \in \mathbb{R}/2\pi\mathbb{Z}$ for $n < j \leqslant n+m$. If $X = (x_1, ..., x_{n+m})$, then

$$\rho(X) = \sum_{i=1}^{n+m} x_i \frac{\partial}{\partial t_i} \tag{7.15}$$

with the coordinates x_i t-independent. (For non-abelian Lie groups, the coordinates x_i will be functions of t.)

This is a hermitian representation acting on the Hilbert space $\mathscr{H}$. The dense domain $D(\rho)$ is the image of $C_c^\infty(B)$ under the quotient mapping

$$C_c^\infty(B) \to C_c^\infty(B)/\{\varphi : (\varphi, \varphi) = 0\}.$$

It is assumed that ρ is extendable: There is a unitary representation $(U, \tilde{\mathscr{H}})$ of G, and an isometry S: $\mathscr{H} \to \tilde{\mathscr{H}}$, such that

$$dU(X)\, S\varphi = S\rho(X)\varphi \qquad \text{for} \quad X \in \mathfrak{g},\ \varphi \in D(\rho). \tag{7.16}$$

Apply the SNAG-theorem [27] to the representation U. We get a projection valued spectral measure E on $\hat{G}$ such that

$$U(a) = \int_{\hat{G}} \langle a, \xi \rangle \, dE(\xi) \qquad \text{for all} \quad a \in G. \tag{7.17}$$

For Borel subsets $\Delta \subset \hat{G}$, the projection $E(\Delta)$ is acting on the (dilated) Hilbert space $\tilde{\mathscr{H}}$.

We shall view the isometry $S\colon \mathscr{H} \to \tilde{\mathscr{H}}$ as a continuous linear mapping in the space $C_c^\infty(B)$, where $C_c^\infty(B)$ is equipped with the natural inductive limit (LF) topology of L. Schwartz [42].

If $\tilde{\mathscr{H}}$ is integrally decomposed according to the spectral multiplicity of the measure E, we obtain

$$\tilde{\mathscr{H}} = \int_{\hat{G}}^{\oplus} \mathscr{H}(\xi)\, d\nu(\xi) \tag{7.18}$$

for a finite positive Radon measure $d\nu$ on $\hat{G}$. This is now a scalar measure. Since $C_c^\infty(B)$ is a nuclear space, there is a measurable family of vector valued distributions S_ξ, taking values in $\tilde{\mathscr{H}}(\xi)$, such that

$$S(\varphi)(\xi) = S_\xi(\varphi) \in \tilde{\mathscr{H}}(\xi) \qquad \text{for} \quad \nu\text{-almost all } \xi \in \hat{G}. \tag{7.19}$$

But we also have

$$S(\rho(X)\varphi) = dU(X)\, S(\varphi), \qquad X \in \mathfrak{g},\ \varphi \in C_c^\infty(B), \tag{7.20}$$

and

$$dU(X)\, S_\xi(\varphi) = \sqrt{-1}\, \langle X, \xi \rangle\, S_\xi(\varphi) \qquad \text{for all} \quad \xi \in \hat{G}. \tag{7.21}$$

Putting together formulas (7.15)–(7.21), we conclude that the distributions $\{S_\xi\}_{\xi \in \hat{G}}$ solve, in the weak sense, an elliptic system of first-order partial differential equations determined by the differential operators $\rho(X)$ from (7.15). It follows that there is a measurable field of vectors $v(\xi)$ such that

$$S_\xi(\varphi) = \left(\int_B \varphi(t) e^{-it\cdot\xi}\, dt \right) v(\xi).$$

The interpretation of $e^{-it\cdot\xi}$ is relative to the chosen coordinate system.

We may regard $C_c^\infty(B)$ as a subspace of $C_c^\infty(G) = \mathscr{D}(G)$. If $\varphi \in C_c^\infty(B)$, then

$$\int_B \varphi(t) e^{-it\cdot\xi}\, dt = \int_G \varphi(t)\, \overline{\langle t, \xi \rangle}\, dt = \hat{\varphi}(\xi),\ \xi \in \hat{G},$$

with $\hat{\varphi}$ denoting the familiar Fourier transform. We conclude that

$$S_\xi(\varphi) = \hat{\varphi}(\xi)\, v(\xi)$$

and

$$(\varphi, \psi) = (S\varphi, S\psi)$$
$$= \int_{\hat{G}} (S_\xi(\varphi), S_\xi(\psi)) \cdot dv(\xi)$$
$$= \int_{\hat{G}} \hat{\varphi}(\xi) \overline{\hat{\psi}(\xi)} \, \|v(\xi)\|^2 \, dv(\xi) \qquad \text{for all} \quad \varphi, \psi \in C_c^\infty(B). \tag{7.22}$$

We also have

$$(S\varphi, S\psi) = \int_{\hat{G}} (dE(\xi) \, S\varphi, S\psi),$$

which means that

$$(S^* E(d\xi) \, S\varphi, \psi) = \hat{\varphi}(\xi) \overline{\hat{\psi}(\xi)} \, \|v(\xi)\|^2 \, dv(\xi).$$

Since

$$(\varphi, \psi) = \int_G \chi_B(t) \, F(t)(\varphi \times \psi^*)(t) \, dt$$
$$= \int_{\hat{G}} (\chi_B F)^\# \, (\xi) \, \hat{\varphi}(\xi) \overline{\hat{\psi}(\xi)} \, d\xi \tag{7.23}$$

by the Plancherel theorem, we obtain the desired extension of F. We have denoted the indicator function of B by χ_B. The transform $h \to h^\#$ is the composition of the Fourier transform with $h(t) \to \overline{h(-t)}$.

Note that we may assume that the spaces $\tilde{\mathscr{H}}(\xi)$ in the decomposition (7.18) are of dimension zero or one. Dimension one corresponds to points in the support of the measure v. If, one particular extension $(U, \tilde{\mathscr{H}})$, the case $\dim \tilde{\mathscr{H}}(\xi) > 1$, occurs, then $\tilde{\mathscr{H}}(\xi)$ may be replaced by a one-dimensional space. The effect of this modification is to change a given extension into a minimal one. Define

$$F(t) = \int_{\hat{G}} \langle t, \xi \rangle \, \|v(\xi)\|^2 \, dv(\xi) \tag{7.24}$$

where the transform is in the sense of distributions. The given continuous function F on C may also be viewed as a distribution, and it follows, from (7.22) and (7.23), that $\tilde{F}$ is an extension of F.

It follows, from (7.17) and (7.23), that

$$(S^* U(a) \, S\varphi, \psi) = \iint \tilde{F}(s + a - t) \, \varphi(s) \, \overline{\psi(t)} \, ds \, dt$$

for all $a \in G$, φ, $\psi \in C_c^\infty(B)$. We conclude that $\tilde{F}$ is a well-defined distribution, concluding the proof of case 1.

Proof of Proposition 7.6. (a) We shall assume that $G \simeq \mathbb{R}^n \times \mathbb{T}^m$. The Lie algebra $\mathfrak{g}$ is $\mathbb{R}^{n+m}$, and the enveloping algebra $\mathfrak{A}$ may be identified with complex polynomials in $n+m$ variables. Let $P \in \mathfrak{A}$, and let $D = (\partial/\partial t_1, ..., \partial/\partial t_{n+m})$. Then

$$\rho(P) = P(D). \tag{7.25}$$

For $\xi \in \hat{G}$, we shall define the symbol $P(\xi)$ by substitution of ξ into the polynomial.

Let $(U, \tilde{\mathscr{H}})$ be an extension of the representation ρ with isometry $S\colon \mathscr{H} \to \tilde{\mathscr{H}}$, satisfying (7.27), and let E be the projection valued spectral measure, obtained by applying the SNAG theorem to the representation U, cf. identity (7.20).

Let $P \in \mathfrak{A}$ be given and assume

$$P(\xi) \geqslant 0 \qquad \text{for all} \quad \xi \in \hat{G}. \tag{7.26}$$

Then

$$\begin{aligned}
&\int_B \int_B F(s-t)\, P(D)\, \varphi(s)\, \overline{\varphi(t)}\, ds\, dt \\
&\quad = (\rho(P)\varphi, \varphi) = (S\rho(P)\varphi, S\varphi)_{\tilde{\mathscr{H}}} \\
&\quad = (dU(P)\, S\varphi, S\varphi)_{\tilde{\mathscr{H}}} = \int_{\hat{G}} P(\xi)\, \|dE(\xi)\, S\varphi\|^2 \geqslant 0
\end{aligned}$$

for all $\varphi \in C_c^\infty(B)$ by virtue of condition (7.26).

This proves one implication of the Proposition 7.6. The converse implication follows from Powers' theorem [37, Corollary 5.5, p. 291].

(b) We showed that minimal extensions $(U, \tilde{\mathscr{H}}, S)$ of $(\rho, \mathscr{H})$ are associated to finite positive Radon measures on $\hat{G}$ such that

$$\begin{aligned}
S\varphi(\xi) &= \hat{\varphi}(\xi), \qquad && \varphi \in C_c^\infty(B) \\
U(a)\, f(\xi) &= \langle a, \xi \rangle\, f(\xi), \qquad && a \in G,\ \xi \in \hat{G}
\end{aligned}$$

and

$$\|\varphi\|_{\mathscr{H}}^2 = \|\hat{\varphi}\|_{\mathscr{L}^2(\hat{G}, \mu)}^2 = \int_{\hat{G}} |\hat{\varphi}(\xi)|^2\, d\mu(\xi),$$

where $\hat{\varphi}$ denotes the Fourier transform of φ.

Indeed, let ν be the measure from formula (7.18) above, and let $\xi \to v(\xi)$ be the measurable field, given by

$$S_\xi(\varphi) = \hat{\varphi}(\xi)\, v(\xi), \qquad \xi \in \hat{G},$$

where $\hat{\varphi}$ denotes the Fourier transform. If we define μ by

$$d\mu(\xi) = |v(\xi)|^2 \, d\nu(\xi),$$

then

$$\|S\varphi\|^2 = \int_{\hat{G}} |\hat{\varphi}(\xi)|^2 \, d\mu(\xi) \tag{7.27}$$

holds for $\varphi \in C_c^\infty(B)$.

It remains to show that $S_\mu(C_c^\infty(B))$ is dense in $\mathscr{L}^2(\hat{G}, \mu)$ if and only if condition (c) in the proposition is satisfied. Let $f \in \mathscr{L}^2(\hat{G}, \mu)$ and assume $(S_\mu \varphi, f) = 0$ for all $\varphi \in C_c^\infty(B)$. Let $\mathscr{F}_\mu f(t) = \int_{\hat{G}} \langle t, \xi \rangle f(\xi)\, d\mu(\xi)$ be defined for $t \in G$. Using Fubini's theorem, we get $\int_B \varphi(t)\, \mathscr{F}_\mu(t)\, dt = 0$ for all $\varphi \in C_c^\infty(B)$. The conclusion in the proposition is immediate from this.

We conclude by noting that the proof carries over *mutatis mutandis* to the case in Theorem 7.3 when F is a given positive definite distribution and the associated representation ρ of the Lie algebra is assumed to be extendable.

EXAMPLE 7.7. Let $G = \mathbb{R}^n$, and let Γ be a lattice, i.e., a discrete subgroup of $\mathbb{R}^n$ such that the quotient $\mathbb{R}^n/\Gamma$ is compact. Let Ω be an open connected fundamental domain for Γ, e.g., an n-dimensional cube. Let $|\Omega|$ denote the Lebesgue measure of the set Ω, and let

$$\Gamma^0 = \{\xi \in \mathbb{R}^n : \xi \cdot \gamma \in 2\pi\mathbb{Z}\}$$

be the dual lattice. For $t \in \mathbb{R}^n$, let

$$\langle t, \lambda \rangle := \prod_i e^{\sqrt{-1}\, t_i \lambda_i} = e^{\sqrt{-1}\, t \cdot \lambda}.$$

By the Poisson summation formula, we have

$$\sum_{\gamma \in \Gamma} \delta(t - \gamma) = |\Omega|^{-1} \sum_{\xi \in \Gamma^0} \langle t, \xi \rangle,$$

or, equivalently,

$$\sum_{\gamma \in \Gamma} \varphi(\gamma) = |\Omega|^{-1} \sum_{\xi \in \Gamma^0} \hat{\varphi}(\xi), \tag{7.28}$$

where δ denotes the Dirac delta function and $\hat{\varphi}$ is the Fourier transform of the test function φ on $\mathbb{R}^n$.

Then we may choose generators $v_1, ..., v_n$ for Γ and integers $l_1, ..., l_n$ such that the subset $\Gamma_{(l)} \subset \Gamma$, defined by

$$\Gamma_{(l)} = \left\{ \sum_{i=1}^{n} k_i v_i : k_i \in \{l_i, l_i + 1\} \right\} \tag{7.29}$$

has the following property: $\Gamma_{(l)}$ is contained in the closure $\bar{\Omega}$, but, if $\gamma = \sum_i k_i v_i$ with some $k_i = l_i$, then the two points γ and γ^*, obtained from γ by replacing k_i with $k_i + 1$, are not both in Ω.

We define

$$F(t) = \sum_{\gamma \in \Gamma_{(l)}} \delta(t - \gamma) \tag{7.30}$$

for $t \in \Omega - \Omega$. Then

$$F(\varphi \times \varphi^*) = \int_{\Omega} |\varphi(t)|^2 \, dt, \tag{7.31}$$

and the extension problem is the one studied by Fuglede [9].

The distribution F has an extension to a distribution $\tilde{F}$ which is given by the same sum, now extended over all of the lattice Γ.

The extended distribution is positive definite, by virtue of the Poisson summation formula, since

$$\sum_{\gamma \in \Gamma} (\varphi \times \varphi^*)(\gamma) = |\Omega|^{-1} \sum_{\xi \in \Gamma^0} |\hat{\varphi}(\xi)|^2;$$

and the unitary representation U, discussed in Theorem 7.3; is obtained by the choice of periodic boundary conditions. The support for the spectral measure of this representation coincides with the dual lattice Γ^0.

This example was discussed earlier in [9], in the context of extensions of operators. There are examples, see [9, 25], where the support of the spectral measure is not a group.

In this example, the "extended" Hilbert space may be taken to *coincide* with the original Hilbert space. The next results give a necessary and sufficient condition for this to happen. It generalizes earlier results in [9, 34] and summarizes our present results for abelian Lie groups:

COROLLARY 7.8. *Let B be an open connected set containing* 0 *in an abelain Lie group G, and let F be a continuous positive definite function on the set $B - B$. Let $\mathscr{H}$ be the Hilbert space obtained from the* GNS-*represen-*

tation applied to F and let ρ be the infinitesimal representation of the enveloping algebra.

Then the following are equivalent:

(1) *ρ has an extension (dilation) $(U, \tilde{\mathscr{H}})$ in some Hilbert space $\tilde{\mathscr{H}}$.*

(2) *There is a* finite *positive Radon measure μ on $\hat{G}$ satisfying the identity:*

$$\int_B \int_B F(s-t)\, \varphi(s)\, \overline{\varphi(t)}\, ds\, dt = \int_{\hat{G}} |\hat{\varphi}(\xi)|^2\, d\mu(\xi) \tag{7.32}$$

for all $\varphi \in C_c^\infty(B)$.

(3) *F has a* continuous *positive definite extension to G.*

(4) *The inequality*

$$\int_B \int_B F(s-t)\, \rho(P)\, \varphi(s)\, \overline{\varphi(t)}\, ds\, dt \geqslant 0 \tag{7.33}$$

holds for all $\varphi \in C_c^\infty(B)$ and all $P \in \mathfrak{A}_+$.

Proof. (1)$\Rightarrow$(2) The assumption in (1) yields a strongly continuous unitary representation U of G in the Hilbert space $\tilde{\mathscr{H}}$ such that

$$\rho(X)\, \varphi(a) = dU(X)\, \varphi(a) = \frac{d}{dt}\bigg|_{t=0} U(\exp tX)\, \varphi(a) \tag{7.34}$$

for all $X \in \mathfrak{g}$, $\varphi \in C_c^\infty(B)$, and $a \in B$.

We have the direct integral decomposition

$$U(a) = \int_{\hat{G}} \langle a, \xi \rangle\, dE(\xi), \qquad a \in G \tag{7.35}$$

and

$$\tilde{\mathscr{H}} = \int_{\hat{G}}^{\oplus} \tilde{\mathscr{H}}(\xi)\, d\nu(\xi),$$

where $\tilde{\mathscr{H}}(\xi)$ is one-dimensional, for ξ in the support of the projection valued measure E, associated to the representation U.

The isometric inclusion (denoted by S) of $C_c^\infty(B)$ into $\tilde{\mathscr{H}}$ is continuous, and it follows, as in the proof of Theorem 7.3 (Part 1) (when $\tilde{\mathscr{H}}$ is chosen minimal) that

$$S_\xi(\varphi) = c(\xi)\, \hat{\varphi}(\xi), \qquad \xi \in G, \tag{7.36}$$

where $\xi \rightarrow c(\xi)$ is scalar valued and measurable on $\hat{G}$.

We conclude that

$$(dE(\xi)\varphi, \varphi) = |c(\xi)|^2 \, |\hat{\varphi}(\xi)|^2 \, d\nu(\xi).$$

and that

$$F(s-t) = \int_{\hat{G}} \langle s-t, \xi \rangle \, |c(\xi)|^2 \, d\nu(\xi).$$

If we define

$$\tilde{F}(t) = \int_{\hat{G}} \langle t, \xi \rangle \, d\mu(\xi) \tag{7.37}$$

and

$$d\mu(\xi) = |c(\xi)|^2 \, d\nu(\xi), \qquad \xi \in \hat{G}, \tag{7.38}$$

it follows that $\tilde{F}$ is a continuous positive definite extension of F. The measure $d\mu(\xi)$ is finite because $F(0) = \int_{\hat{G}} d\mu(\xi)$ is finite.

We have verified (2) and (3) in one stroke, since the norm in $\mathscr{H}$ is defined by

$$\|\varphi\|^2 = \int_B \int_B F(s-t) \, \varphi(s) \, \overline{\varphi(t)} \, ds \, dt$$

and the remaining conclusion in (2) is immediate from Fubini's theorem.

(3) $\Rightarrow$ (2) Assume $\tilde{F}$ is a continuous positive definite extension of F, and let $d\mu$ be the positive finite Radon measure on $\hat{G}$ which exists by Bochner's theorem and satisfies

$$\tilde{F}(t) = \int_{\hat{G}} \langle t, \xi \rangle \, d\mu(\xi), \qquad t \in G.$$

Then a substitution into formula (7.39), and a second application of Fubini's theorem yields the conclusion in (2).

(2) $\Rightarrow$ (1) We may define a projection valued measure E on the Hilbert space $\mathscr{L}^2(\hat{G}, \mu)$ by

$$(dE(\xi) f_1, f_2) = f_1(\xi) \, \overline{f_2(\xi)} \, d\mu(\xi). \tag{7.39}$$

This is a projection valued, orthogonal measure on the Hilbert space $\mathscr{L}^2(\hat{G}, \mu)$ if (2) is assumed, and the corresponding unitary representation U of G is determined by

$$U(a) \, f(\xi) = \langle a, \xi \rangle \, f(\xi) \qquad \text{for} \quad a \in G, \, \xi \in \hat{G}, \, f \in \mathscr{L}^2(\hat{G}, \mu). \tag{7.40}$$

The compression of the projection valued measure E to $\mathscr{H}$ is $S^*E(\cdot)S$, and it is determined by the formula

$$(dE(\xi)\, S\varphi, S\varphi) = |\hat{\varphi}(\xi)|^2\, d\mu(\xi).$$

It remains to show that

$$\rho(P)\varphi = dU(P)\varphi \qquad \text{for all} \quad P \in \mathfrak{A},\ \varphi \in C_c^\infty(B).$$

We have

$$\begin{aligned}
(dU(P)\varphi, \psi) &= \int_{\hat{G}} P(\xi)(dE(\xi)\varphi, \psi) \\
&= \int_{\hat{G}} P(\xi)\, \hat{\varphi}(\xi)\, \overline{\hat{\psi}(\xi)}\, d\mu(\xi) \\
&= \int_{\hat{G}} (\rho(P)\varphi)^\wedge\ (\xi)\, \overline{\hat{\psi}(\xi)}\, d\mu(\xi) \\
&= \int_B \int_B F(s-t)\, \rho(P)\, \varphi(s)\, \overline{\psi(t)}\, ds\, dt \\
&= (\rho(P)\varphi, \psi)
\end{aligned}$$

for all $P \in \mathfrak{A}$ and φ, $\psi \in C_c^\infty(B)$, where we used (2) and the elementary formula

$$(\rho(P)\varphi)^\wedge\ (\xi) = P(\xi)\, \hat{\varphi}(\xi), \qquad \xi \in \hat{G}$$

for the Fourier transform of a constant coefficient partial differential operator.

The equivalence to condition (4) is proved in detail in Proposition 7.6 above.

Proof of Theorem 7.3. *Case* 2. This is the case where G is the Heisenberg group and F is a given positive definite continuous function on a set of the form $B \cdot B^{-1}$, as stated in the theorem. We shall take F to be a positive definite distribution, since the proof is essentially the same. It will only be sketched since the details are similar to the proofs above, and in Section 5.

We let $\mathscr{H}$ be the Hilbert space, defined from $C_c^\infty(B)$ with the inner product given by $F(\varphi \times \psi^*)$. We have an associated representation ρ of the Lie algebra $\mathfrak{g}$, and we assume that ρ is extendable. We pick a minimal extension $(U, \tilde{\mathscr{H}})$ with an isometry S: $\mathscr{H} \to \tilde{\mathscr{H}}$ such that

$$dU(X)\, S(\varphi) = S(\rho(X)\varphi) \qquad \text{for} \quad X \in \mathfrak{g},\ \varphi \in C_c^\infty(B). \tag{7.41}$$

Let $X_3 \neq 0$ be a central element in $\mathfrak{g}$, and consider the spectral resolution for the operator $dU(X_3)$

$$\tilde{\mathscr{H}} = \int^{\oplus} \tilde{\mathscr{H}}(\xi)\, d\nu(\xi) \tag{7.42}$$

with a finite positive Radon measure on the spectrum. We get a corresponding decomposition

$$U = \int^{\oplus} U^{\xi}\, d\nu(\xi) \tag{7.43}$$

with the component representation U^{ξ} acting on the Hilbert space $\mathscr{H}(\xi)$. Since $(U, \tilde{\mathscr{H}})$ is minimal, we may take U^{ξ} to be irreducible similarly to the choice in the case when G was assumed abelian. By the Stone–von Neumann uniqueness theorem [18], we may assume that U^{ξ} is the Schrödinger representation with frequency $|\xi|$. We may identify $\tilde{\mathscr{H}}(\xi)$ with $\mathscr{L}^2(\mathbb{R})$ so that

$$dU^{\xi}(L) = \frac{1}{2}\left(-\left(\frac{d}{dx}\right)^2 + (\xi x)^2\right) \tag{7.44}$$

is the harmonic oscillator Hamiltonian when dU^{ξ} is evaluated on the sub-Laplacian L.

For the inner product, we have

$$\begin{aligned}(\varphi, \psi) &= F(\varphi \times \psi^*) = (S\varphi, S\psi)\\ &= \int_{\mathbb{R}} \operatorname{trace}(U^{\xi}(\varphi)\, U^{\xi}(\psi)^*)\, |\xi|\, d\mu(\xi)\end{aligned} \tag{7.45}$$

for some measure μ, mutually absolutely continuous to ν, by the Plancherel theorem for G [35].

The operator $U^{\xi}(\varphi)$ is defined by

$$U^{\xi}(\varphi) = \int_G \varphi(g)\, U^{\xi}(g^{-1})\, dg$$

for $\varphi \in C_c^{\infty}(B)$. Note that $U^{\xi}(\varphi)$ is defined also for $\varphi \in C_c^{\infty}(G)$.

It can be shown, using (7.42), as in the proof of Case 1 and the proof of Theorem 7.1, that the expression on the right-hand side of (7.45) is defined for all $\varphi, \psi \in C_c^{\infty}(G)$. It is a distribution in both variables. If $\tilde{F}$ denotes the kernel, we show translation invariance:

$$\tilde{F}(g_1, g_2) = \tilde{F}(g_1 \cdot a, g_2 \cdot a), \qquad a \in G.$$

It follows that there is a distribution on G, also denoted $\tilde{F}$, such that $\tilde{F}(g_1, g_2) = \tilde{F}(g_1 \cdot g_2^{-1})$. When it is substituted back into formula (7.45) we conclude that the distribution $\tilde{F}$ extends the originally given distribution F.

Note added in proof. In Theorem 5.3, the given kernel function K was assumed analytic. This assumption has now been removed by the author, and the details will appear separately. We then use this result to obtain the converse implication in Theorem 7.1 above, showing that extendability of the given (locally defined) function is equivalent to the existence of a unitary dilation for the associated representation ρ.

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